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# COST ALLOCATION IN COMBINATORIAL OPTIMIZATION PROBLEMS - THE TSP CASE

by

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## ABSTRACT

In this paper we examine the issue of a "proper" cost allocation in a combinatorial problem such as the travelling salesman problem - (TSP). What share of a total cost for an optimal or nonoptimal solution to a TSP should be attributed to an individual node-customer? The total cost of a TSP tour has to be covered by the participants, and this paper addresses the question of how should this cost be split among the individual customers. Given several criterias of a "proper" cost allocation, the widly used economic principle of "cost plus" charge emerges as the common mathematical solution for cost allocation in this type of combinatorial problems. Each customer should be charged a fraction of the total cost which equals to his incremental contribution to the cost plus a fixed amount which is the same for all customers.

## 1. INTRODUCTION

The Travelling Salesman Problem - (TSP) is the problem of constructing the shortest simple circuit passing through a given set of n connected nodes. It is perhaps the single most intensively studied combinatorial problem in the last thirty-five years. A vast literature on the TSP presently exists and more is added every year. Almost all aspects of this problem have been thoroughly examined and discussed. A recent book of Lawler et al. (1985) illustrates the preoccupation with this problem and its derivatives. This paper does not address the TSP directly, but examines a derivative of the TSP and (as in a great deal of TSP related research) it extands this study to other related combinatorial problems.

This paper is concerned with "cost" allocation in TSP solutions. More precisely, given an optimal solution of a TSP, what share of the total length of this solution should be attributed to an individual element-node of the problem? The present body of literature on the TSP does not, to the authors knowledge, examine this specific aspect of the problem. The implications of the cost allocation solution in the TSP case are of significant practical importance. Very similar questions concerning the "proper" cost allocation among elements of a combinatorial problem arise in the machine scheduling environment, and many other examples exist. An attempt is made in this paper to sketch appropriate extensions of the results presented here to some other combinatorial optimization problems.

## 2. THE TSP CASE

It is well known that dividing the total "cost" or distance of an optimal TSP solution-tour equally among the n nodes of that tour may completely distort the "true" contribution of a particular node to the total cost of the tour. If most of the nodes are bunched up together closely and a sigle node is isolated far from this main concentration then it is clearly unfair, or inappropriate, to allocate the same cost to the isolated node as to a node in the main dense cluster of nodes. In this illustration of the problem, we implicitly assume a metric which measures the "cost" (or distance) of successively visiting nodes in the set of nodes through which a TSP tour is to be constructed. Dividing the cost of a tour equally among its elements has the attractive property of the sum, i.e., that it adds up exactly to the total cost of the tour. This property is an essential property of an appropriate cost allocation among the various elements of a task such as a TSP tour. At this point, we introduce the definition of the incremental cost for a node i in the TSP tour.

### 2.1 Exact incremental cost

Denote the incremental cost for a node i in the TSP tour by a(i) and the total cost of an optimal TSP through a given set N of n nodes as L{N}.

Define a(i) as follows :

$$a(i) \equiv L\{N\} - L\{N-i\}.$$

a(i) corresponds to the true incremental cost of adding the node i to the set {N-i} of nodes for the TSP tour.

It is obvious that :

$$L\{N\} \geq \sum_{i=1}^{n} a(i). \qquad (2.1)$$

The interpretation of this inequality is that allocating the total cost of an optimal TSP tour according to the incremental contribution of each of its elements does not constitute a proper cost allocation. Note that in order to calculate the values $L\{N\}$ and $L\{N-i\}$ for each i, n+1 TSP problems have to be solved to optimality, i.e., in order to calculate the incremental values $a(i)$, $1<i<n$, n+1 instances of na NP-hard problem have to be solved optimally.

Denote by $b(i)$, the cost allocated to element i. In order for $b(i)$, $1<i<n$, to be a proper cost allocation, the following equality must be satisfied :

$$L\{N\} = \sum_{i=1}^{n} b(i)$$

Assume that two sets of n numbers are given : $\{\tilde{b}(i):1<i<n\}$ and $\{\bar{b}(i):1<i<n\}$. Both sets of numbers satisfy the proper cost allocation equality:

$$L\{N\} = \sum_{i=1}^{n} \tilde{b}(i) = \sum_{i=1}^{n} \bar{b}(i).$$

How can these two sets of numbers be compared in terms of an adequate cost allocation? There could be a number of equally acceptable criteria for the scale differentiation of cost allocations.

We introduce two such criteria CA1 and CA2. Both derive a proper cost allocation for the TSP by solving an optimization problem. Furthermore, as we shall see, these two criteria leed to the same solution in some cases.

(CA1)

$$\text{Min} \{\underset{i}{\text{Max}}\{|b(i)-a(i)|\}\}$$

$$\text{subject to} \sum_{i=1}^{n} b(i) = L\{N\}$$

$$b(i) > 0, \qquad 1<i<n.$$

CA1 attempts to allocate the cost of an optimal TSP tour amongst the different elements with respect to the maximal deviation of any of these costs from the element's incremental cost.

(CA2)

$$\text{Min} \sum_{i=1}^{n} (b(i)-a(i))^{2k}$$

$$\text{subject to} \sum_{i=1}^{n} b(i) = L\{N\}$$

$$b(i) > 0, \qquad 1<i<n$$

where k could be any positive integer.

A more restricted equivalent version of CA1 can be obtained if the TSP is solved in euclidian space.

(CA1')

$$\text{Min} \{\underset{i}{\text{Max}}\{b(i)-a(i)\}\}$$

$$\text{subject to} \sum_{i=1}^{n} b(i) = L\{N\}$$

$$b(i) > a(i) > 0, \qquad 1<i<n.$$

Theorem 1 : In a Euclidian space CA1 and CA1' are equivalent.

Proof : It is clear that in Euclidian space $a(i) > 0$, $1<i<n$. It remains to prove that $\bar{b}(i) > a(i)$ for $1<i<n$ given that $\bar{b} = [\bar{b}(1),\bar{b}(2),\ldots,\bar{b}(n)]^T$ is a solution to CA1.

This can be proved by contradiction. Assume $\bar{b}=[\bar{b}(1),\bar{b}(2),\ldots,\bar{b}(n)]^T$ is an optimal solution of CA1 and for some j, $1\leq j\leq n$, $a(j) > \bar{b}(j)$. Furthermore, denote

$$I_1 = \{i : a(i)-\bar{b}(i) > 0\}$$

$$I_2 = \{j : |\bar{b}(j)-a(j)| = \max_{1\leq i\leq n} \{|\bar{b}(i)-a(i)|\}=\gamma\}$$

$$\beta = \min_{i\in I_1} \{a(i)-\bar{b}(i)\}$$

Several cases have to be studied.

Case 1. Suppose $I_1 \cap I_2 = \emptyset$. Define a new solution $\tilde{b}$ as follows :

$$\tilde{b}(i) = \begin{cases} \bar{b}(i)-\varepsilon & \text{if } i\in I_2 \\ \bar{b}(i)+k\varepsilon & \text{if } i\in I_1 \\ \bar{b}(i) & \text{otherwise} \end{cases}$$

where $k = \dfrac{|I_2|}{|I_1|}$ and $0<\varepsilon \leq \min \{\gamma, \dfrac{|I_1|}{|I_2|}\beta\}$ .

Then, $\sum_{i=1}^{n} \tilde{b}(i) = \sum_{i=1}^{n} \bar{b}(i) - |I_2|\,\varepsilon + \dfrac{|I_2|}{|I_1|}\cdot|I_1|\,\varepsilon = L(N)$,

and

$\tilde{b}(i) > a(i)$ for $i\in I_2$ because $\varepsilon < \gamma$. Also since $\varepsilon \leq \dfrac{|I_1|}{|I_2|}\beta$, then $k\varepsilon \leq \beta$ and for all $i\in I_1$,

$$a(i)-\tilde{b}(i) = a(i)-\bar{b}(i)-k\varepsilon \geq \beta-\beta=0.$$

Thus for $i\in I_1$,

$$|\tilde{b}(i)-a(i)| = a(i)-\tilde{b}(i) = a(i)-\bar{b}(i)-k\varepsilon < a(i)-\bar{b}(i) = |\bar{b}(i)-a(i)|.$$

It follows that

$$\underset{1\le i\le n}{\text{Max}}\ \{|\tilde{b}(i)-a(i)|\} < \underset{1\le i\le n}{\text{Max}}\ \{|\bar{b}(i)-a(i)|\}.$$

Case 2. Suppose $I_1 \supset I_2$. Define a new solution to $\tilde{b}$ as follows :

$$\tilde{b}(i) = \begin{cases} \bar{b}(i)-\varepsilon & \text{if } i=\bar{i} \\ \bar{b}(i)+k\varepsilon & \text{if } i\in I_1 \\ \bar{b}(i) & \text{otherwise} \end{cases}$$

where $\bar{i} \notin I_1$, $k = \dfrac{1}{|I_1|}$, and $0 < \varepsilon < \min\{|I_1|\beta,\ \bar{b}(\bar{i})-a(\bar{i})\}$.

Then $\sum_{i=1}^{n} \tilde{b}(i) = \sum_{i=1}^{n} \bar{b}(i) - \varepsilon + |I_1| \cdot \dfrac{\varepsilon}{|I_1|} = L(N)$,

and, because $\varepsilon < \bar{b}(\bar{i})-a(\bar{i})$,

$\tilde{b}(i) > 0$ for all $i$, $1\le i\le n$.

Furthermore, as in Case 1, $|\tilde{b}(i)-a(i)| < |\bar{b}(i)-a(i)|$ for all $i\in I_1$.

Hence $\underset{1\le i\le n}{\text{Max}}\ \{|\tilde{b}(i)-a(i)|\} < \underset{1\le i\le n}{\text{Max}}\ \{|\bar{b}(i)-a(i)|\}$.

Case 3. Suppose $I_1 \cap I_2 \neq \emptyset$ and $I_2 \setminus (I_1 \cap I_2) \neq \emptyset$.

Define a new solution $\tilde{b}$ as follows :

$$\tilde{b}(i) = \begin{cases} \bar{b}(i)-\varepsilon & \text{if } i\in I_2\setminus(I_1\cap I_2) \\ \bar{b}(i)+k\varepsilon & \text{if } i\in I_1 \\ \bar{b}(i) & \text{otherwise} \end{cases}$$

where $k = \dfrac{|I_2\setminus(I_1\cap I_2)|}{|I_1|}$, $0 < \varepsilon < \text{Min}\{\gamma,\ \dfrac{|I_1|}{|I_2\setminus(I_1\cap I_2)|}\beta\}$.

Then $\sum_{i=1}^{n} \tilde{b}(i) = \sum_{i=1}^{n} \bar{b}(i) - |I_2 \setminus (I_1 \cap I_2)|\,\varepsilon + \frac{|I_2 \setminus (I_1 \cap I_2)|}{|I_1|} \cdot |I_1|\,\varepsilon = L(N).$

As in Case 1, it is easy to verify that

$$\tilde{b}(i) > a(i) \qquad \text{for } i \in I_2 \setminus (I_1\ I_2)$$

$$|\tilde{b}(i)-a(i)| < |\bar{b}(i)-a(i)| \qquad \text{for } i \in I_1 .$$

Hence $\underset{1 \le i \le n}{\text{Max}} \{|\tilde{b}(i)-a(i)|\} < \underset{1 \le i \le n}{\text{Max}} \{|\bar{b}(i)-a(i)|\}$ .

In all cases, a new solution having a smaller value can be constructed which contradicts the assumption that $\bar{b}$ is an optimal solution to CA1. □

In Theorem 1 the Euclidian space can be replaced with the requirement that all a(i) be nonnegative.

Consider problem (CA2) with k=1. The problem takes the form

$$\text{Min} \sum_{i=1}^{n} \tfrac{1}{2}(b(i)-a(i))^2 \equiv \text{Min} \sum_{i=1}^{n} \tfrac{1}{2}b(i)^2 - a(i)b(i)$$

(CA2')

$$\text{Subject to} \sum_{i=1}^{n} b(i) = L(N)$$

$$b(i) > 0, \qquad 1 \le i \le n.$$

It follows from the results in Ferland et al. (1978) and Helgason et al. (1980) that an optimal solution for this problem is identified as follows:

i) assume that

$$a(n) \le a(n-1) \le \ldots \le a(2) \le a(1);$$

ii) for $1 \le k \le n$, let

$$\lambda_k = \frac{\sum_{i=1}^{k} a(i) - L(N)}{k};$$

iii) let $\tau$ be the largest index k such that

$$\lambda_k \le a_k ;$$

then $b(i) = \text{Max}\{a(i)-\lambda_\tau, 0\}, \quad 1 \le i \le n$ (2.2)

is an optimal solution for (CA2').

From (2.1), it follows that $\lambda_n \le 0$. Since $a(n) > 0$, it follows that $\bar{b}$ is an optimal solution of (CA2') where

$$\bar{b}(i) = a(i) + \Delta , \quad 1 \le i \le n$$

and

$$\Delta = \frac{L(N) - \sum_{i=1}^{n} a(i)}{n}.$$

Theorem 2. The solution

$$\bar{b}(i) = a(i) + \Delta, \quad 1 \le i \le n$$

with $\Delta = \dfrac{L(N) - \sum_{i=1}^{n} a(i)}{n}$ is an optimal solution of problem (CA1).

Proof. It is easy to see that $\bar{b}$ is a feasible solution of (CA1'). Reformulate (CA1') as

$$\text{Min } f(b)$$

$$\text{Subject to } \sum_{i=1}^{n} b(i) = L(N)$$

$$b(i) \ge a(i) > 0, \quad 1 \le i \le n$$

where $f(b) = \underset{1 \le i \le n}{\text{Max}} \{b(i)-a(i)\}$. Now

$$f(\bar{b}) = \frac{L(N) - \sum_{i=1}^{n} a(i)}{n} = \bar{b}(i) - a(i), \qquad 1 \le i \le n.$$

Consider any other feasible solution of (CA1'). Since $\sum_{i=1}^{n} b(i) = \sum_{i=1}^{n} \bar{b}(i) = L(N)$, it follows that if $b \neq \bar{b}$, then there exists at least one index $\bar{i}$, $1 \le \bar{i} \le n$ such that

$$b(\bar{i}) > \bar{b}(\bar{i}).$$

Hence $f(\bar{b}) = \bar{b}(\bar{i}) - a(\bar{i}) < b(\bar{i}) - a(\bar{i}) \le f(b)$, and $\bar{b}$ is an optimal solution of (CA1'). From Theorem 1, it follows that $\bar{b}$ is an optimal solution of (CA1). □

One interpretation of the optimal cost allocation for the TSP resulting from solving the (CA1') problem is that of the well-known "cost plus" policy. The proper cost allocation involves costing all the element-nodes (customers) according to their incremental contribution a(i) plus some fixed amount in order to cover the cost of the TSP tour.

## 2.2 Approximate incremental cost

In Euclidian space one reasonable approximation of the time incremental cost of element i, a(i) can be obtained in terms of the preceding and succeeding nodes on the TSP tour. Denote these nodes by p and s respectively and the corresponding distances by $c_{pi}$, $c_{is}$ and $c_{ps}$.

The following inequality is always true :

$$0 < a(i) \le c_{pi} + c_{is} - c_{ps} \equiv \bar{a}(i)$$

$\bar{a}(i)$ is an approximation from above of a(i). Consequently, we cannot require that a proper cost allocation b(i) be always greater than or equal to a(i). If a(i) is calculated instead of a(i) for each i, we can easily

find a bound for the "error" in cost resulting from this substitution :

$$0 \leq |b(i)-\bar{a}(i)| \leq \bar{a}(i) - a(i) + (b(i)-a(i)).$$

In other words the "error" is not greater than $\bar{a}(i)-a(i)$.

In the case where $\sum_{i=1}^{n} \bar{a}(i) \leq L\{N\}$, we can find a proper cost allocation with respect to $\bar{a}(i)$, $1\leq i\leq n$, by solving a problem identical to CA1'.

$$\text{Min } \{\text{Max}_i\ b(i)-\bar{a}(i)\}$$

$$\text{subject to } \sum_{i=1}^{n} b(i) = L\{N\}$$

$$b(i) \geq \bar{a}(i) \geq 0, \qquad 1\leq i\leq n.$$

In the case where $\sum_{i=1}^{n} \bar{a}(i) > L\{N\}$, in order to obtain a proper cost allocation the problem must be changed slightly :

(CA1a)

$$\text{Min } \{\text{Max}_i\ \{ |b(i)-\bar{a}(i)| \}\}$$

$$\text{subject to } \sum_{i=1}^{n} b(i) = L\{N\}$$

$$b(i) \geq 0, \qquad 1\leq i\leq n.$$

An optimal solution for (CA1a) is obtained from (2.2) where $a(i)$ is replaced by $\bar{a}(i)$, $1\leq i\leq n$. Indeed, for any other feasible solution $b$, there exists an index $\bar{i}$, $1\leq\bar{i}\leq n$, such that $b(\bar{i}) < \bar{b}(\bar{i})$. From the definition of $\lambda_\tau$ it is easy to verify that $\bar{b}(i) = 0$, for all $i>\tau$. Hence $\bar{i} \leq \tau$. On the other hand, for all $i \leq \tau$, $a(i) - \bar{b}(i) = \lambda_\tau$, and if follows that

$$\lambda_\tau = a(\bar{i}) - \bar{b}(\bar{i}) < a(\bar{i}) - b(\bar{i}).$$

Hence $\underset{1\le i\le n}{\text{Max}} \{|b(i)-a(i)|\} \ge a(\bar{i})-b(\bar{i}) \ge \lambda_\tau = \underset{1\le i\le n}{\text{Max}} \{ \bar{b}(i)-a(i) \}$.

In order to obtain the value for $L\{N\}$ the TSP has to be solved optimally. Since it is an NP-hard problem, for problems with large numbers of elements, this can be prohibitively expensive. If a fast heuristic method is implemented to solve the TSP, the resulting tour has a length denoted by $\tilde{L}\{N\} \ge L\{N\}$. The incremental costs for element-nodes of the tour are defined similarly as before :

$$\tilde{a}(i) = \tilde{L}\{N\} - \tilde{L}\{N-i\}.$$

Since we assume that the heuristic method for the TSP is fast, solving n+1 instances of the TSP is also a rapide process, relatively speaking. Consequently, the $\tilde{a}(i)$ values ($1\le i\le n$) can be computed. The resulting cost allocation problem is identical to CA1' but with the $\tilde{a}(i)$'s taking the place of the $a(i)$'s. In a sense, the $\tilde{a}(i)$'s are the "true" incremental costs since they are computed with the same algorithmic method as the $\tilde{L}\{N\}$ - the TSP tour length.

## 3. MACHINE SCHEDULING PROBLEM

Assume a machine scheduling problem where n is the number of jobs and m is the number of parallel processors. All jobs are available at time zero. If job $J_i$ is processed on machine k after job $J_j$, then a setup of cost of $S^k_{ij}$ is incurred. Given that each job can be processed on any one of the machines, the objective might be to minimize the time to completion of the last job in the system or what is referred to as the makespan. This problem is known to be NP-hard (see Garey et al., 1979) even in the case of one machine where it is equivalent to an open tour TSP. The triangle inequality usually does not hold in this problem nor does symmetry. In

order to allocate the total processing cost to the individual jobs, some sort of a cost allocation problem has to be solved. Since it is implicit that the $S_{ij}^{k}$ are nonnegative, a cost allocation approach identical to that of (CA1') might be appropriate. Because, as indicated, this type of machine scheduling problem is NP-hard, any heuristic algorithm which constructs a feasible solution can be used to allocate the cost properly among the jobs in a manner identical to the heuristic solution of the TSP.

The same approach to cost allocation is applicable to other combinatorial problems for which the sum of the incremental costs of the elements in the solution is less than or equal to the total cost of the system.

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